Numerical Methods - Assignment 3

Due Tuesday, Sep 17

The following is a generalisation of the Sep 10 lecture. You may need to use the same methods of proof, *e.g.* for questions 4-6.

Let $f : [a, b] \to \mathbb{R}$ be a function. We assume it to admit a continuous fourth derivative $f^{(4)}$, and we set

$$M_4 = \|f^{(4)}\|_\infty = \max\{|f^{(4)}(x)|; x \in [a, b]\}.$$

1 - Compute the following integrals by using the substitution $x = \frac{a+b}{2} + t\frac{b-a}{2}$, $t \in [-1, 1]$.

$$\int_a^b (x-a)dx \quad ; \quad \int_a^b (b-x)dx \quad ; \quad \int_a^b (x-a)(b-x)dx \quad ;$$

$$\int_a^b (x-a)(x-b)(x-\frac{a+b}{2})dx \quad ; \quad \int_a^b (x-a)(x-b)(x-\frac{a+b}{2})^2 dx.$$

The Simpson rule

2 - We set $c = (a+b)/2$. Prove that there is a polynomial function g of degree at most 3 such that f and g have the same values at a, b and c, and that $f'(c) = g'(c)$. You may find constants α, β, γ, δ such that

$$g(x) = \alpha(x-a) + \beta(b-x) + \gamma(x-a)(b-x) + \delta(x-a)(b-x)(x-c).$$

3 - We set $S_1(f) = \int_a^b g$. Give an explicit formula for $S_1(f)$, of the form

$$S_1(f) = Af(a) + Cf(c) + Bf(b),$$

where the constant A, B and C depend only on $b-a$.
The use of $S_1(f)$ to estimate the integral $I = \int_a^b f$ is the (one-step) Simpson rule. We will now seek an upper bound for the error term $\mathcal{E}_1(f) = I - S_1(f)$.

4 - We fix $x_0 \in (a, b) \setminus \{c\}$. Prove that there is a polynomial function h of degree at most 4 such that

$$h(a) = h(b) = h(c) = h'(c) = 0 \quad \text{and} \quad h(x_0) = f(x_0) - g(x_0).$$

You may find a constant λ such that $h(x) = \lambda(x-a)(b-x)(x-c)^2$. By iterated use of Rolle's theorem, deduce that

$$|f(x_0) - g(x_0)| \leqslant \frac{M_4}{24}(x_0-a)(b-x_0)(x_0-c)^2.$$

5 - Conclude that $|\mathcal{E}_1(f)| \leqslant \dfrac{M_4(b-a)^5}{2880}$.

The composite Simpson rule

6 - For an integer $n \geqslant 1$, we divide the interval $[a, b]$ into n subintervals $[a_i, a_{i+1}]$ of equal length. We approximate the integral $I = \int_a^b f$ by using the Simpson rule on each of these subintervals. Give a formula for the estimate $S_n(f)$, and a bound for the error term $\mathcal{E}_n(f) = I - S_n(f)$.

7 - Compare the formula for $S_{2n}(f)$ (composite Simpson rule) and that obtained by extrapolation from the trapezoid rule, *i.e.*, $(4T_{2n}(f) - T_n(f))/3$. What can you observe?